MATHEMATICS

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AIM POINTS
MATHEMATICS
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ADJOINT AND INVERSE OF A MATRIX

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. If $A = [a_{ij}]$ is a square matrix of order n and C_{ij} denote the confactor of a_{ij} in A, then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

   i.e., $\text{adj } A = [C_{ij}]^T$

   If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

2. The adjoint of a square matrix of order can be obtained by interchanging the diagonal elements and changing the signs of off–diagonal elements.

   If, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

3. If A is a square matrix of order n, then $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)A$.

4. Following are some properties of adjoint of a square matrix :

   If A and B are square matrices of the same order n, then

   (i) adj (B) = (adj b) (adj A)

   (ii) $\text{adj } A^T = (\text{adj } A)^T$

   (iii) $\text{adj } (\text{adj } A) = |A|^{n-1} A$

   (iv) $|\text{adj } A| = |A|^{n-1}$

5. A square matrix A of order n is invertible of there exists a square matrix B of the same order such that $AB \setminus I_n = BA$.

   In such a case, we say that the inverse of matrix A is B and we write $A^{-1} = B$.

   Following are some properties of inverse of a matrix :

   (i) Every invertible matrix possesses a unique inverse.

   (ii) If A is an invertible matrix, then $(A^{-1})^{-1} = A$

   (iii) A square matrix is invertible iff it is non–singular

   (iv) If A is a non–singular matrix, then

   $$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

   (v) If A and B are two invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$

   (vi) If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$

   (vii) The inverse of an invertible symmetric matrix is a symmetric matrix.

   (viii) If A is a non–singular matrix, then $|A^{-1}| = \frac{1}{|A|}$

6. The following are three operations applied on the rows (columns) of a matrix :

   (i) Interchange of any two rows (columns).

(ii) Multiplying all elements of a row (column), the corresponding elements of any other row (column) multiplied b an scalar.

7. A matrix obtained from an identity matrix by a single elementary operation is called an elementary matrix.

8. Every elementary row (column) operation on an $m \times n$ matrix (not identity matrix) can be obtained by pre–multiplication (post–multiplication) withthe corresponding elementary matrix obtained from the dentity matrix $I_m(I_n)$ by subjecting it to the same elementary row (column) opeation.

9. In order to find the inverse of a non–singular square matrix A by elementary operations, we write

$$A = IA$$

Now we perform a sequence of elementary row operations successively on A on the LHS and the pre–factor I on RHS till we obtain

$$I = BA$$

The matrix B, so obtained, is the desired inverse of matrix A.

EXERCISE-1

1. Every invertible matrix possesses a unique inverse.
2. The inverse of an invertible symmetric matrix is a symmetric matrix.
3. If A is an invertible matrix of order 3 and $|A| = 5$, then find $|\text{adj } A|$.
4. If A and B are non–singular square matrices of the same order, then adj AB = (adj B) (adj A).
5. If A is an invertible square matrix, then $\text{adj } A^T = (\text{adj } A)^T$.
6. Prove that adjoint of a symmetric matrix is also a symmetric matrix.
7. If A is an invertible matrix of order 3×3 such that $|A| = 2$, then find adj (adj A).
8. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, find adj A and verify that $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_3$.
9. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19}A$.
10. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and verify that $A^{-1}A = I_3$.
11. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.
12. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1} A^{-1}$.

13. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = 0$. Hence, find A^{-1}.

14. Show that the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I_3 = O$ and hence find A^{-1}.

15. Given $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Computer $(AB)^{-1}$.

16. Show that : $\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

17. Showt that $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the equation $A^2 + 4A - 42I = 0$. Hence, find A^{-1}.

18. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$. Hence, find A^{-1}.

19. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, verify that $A^2 - 4A + I = O$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

20. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$.

21. Solve the matrix equation $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, where X is a 2 × 2 matrix.

22. Find the matrix X for which $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} X \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$.

23. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11I_3 =)$.

24. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Verify that $A^3 - 6A^2 + 9A + 4I = O$ and hence find A^{-1}.

25. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. Hence, find A^{-1}.

26. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ by using elementary row transformations.

27. If A is a square matrix of order 3 such that $|A| = 5$, write the value of $|\text{adj } A|$.

28. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $(\text{adj } A) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then find the value of k.

29. If A is a square matrix such that $A(\text{adj } A) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then write the value of $|\text{adj } A|$.

30. If $A = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$, then find $|\text{adj } A|$.

EXERCISE-2

1. If $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then adj A is

   (a) $\begin{bmatrix} -d & -b \\ -c & a \end{bmatrix}$ (b) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (c) $\begin{bmatrix} d & b \\ c & a \end{bmatrix}$ (d) $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$

2. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then the value of $|\text{adj } A|$ is

   (a) a^{27} (b) a^9 (c) a^6 (d) a^2

3. If B is a non–singular matrix and A is a square matrix, then det (B–1 AB) is equal to

   (a) $\text{Det}(A^{-1})$ (b) $\text{Det}(B^{-1})$ (c) Det(A) (d) Det(B)

4. If A and B are square matrices such that $B = -A^{-1} BA$, then $(A + B)^2 =$

   (a) O (b) $A^2 + B^2$ (c) $A^2 \mid 2AB + B^2$ (d) None of these

5. The matrix $\begin{bmatrix} 5 & 10 & 3 \\ -2 & -4 & 6 \\ -1 & -2 & b \end{bmatrix}$ is a singular matrix, if the value of b is

(a) –3 (b) 3 (c) 0 (d) non–existent

6. If A is a square matrix such that $A^2 = I$, then A^{-1} is equal to

(a) $A + I$ (b) A (c) 0 (d) 2A